Homological algebra solutions Week 10

Throughout, we will abreviate *Cartan-Eilenberg resolution* as *CE-resolution*.

1. (a) Consider the commutative diagram with exact column

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ \cdots & \longrightarrow & B_p(P,d^h)_1 & \longrightarrow & B_p(P,d^h)_0 & \longrightarrow & \operatorname{Im}(d^A_{n+1}) = B_p(A) \\ & & & & & & \downarrow \\ & & & & & & \ker(d^A_p) = Z_p(A) \\ & & & & & & \downarrow \\ \cdots & \longrightarrow & H_p(P,d^h)_1 & \longrightarrow & H_p(P,d^h)_0 & \longrightarrow & Z_p(A)/B_p(A) = H_p(A) \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

where the rows are projective resolutions of $B_p(A)$ and $H_p(A)$. By the Horseshoe Lemma (Exercise 5.4), we can assemble a projective resolution of $Z_p(A)$

$$B_p(P,d^h)_\bullet \oplus H_p(P,d^h)_\bullet \to Z_p(A)$$

where the right hand column lifts to an exact sequence of chain complexes

$$0 \longrightarrow B_p(P,d^h)_\bullet \longrightarrow B_p(P,d^h)_\bullet \oplus H_p(P,d^h)_\bullet \longrightarrow H_p(P,d^h)_\bullet \longrightarrow 0\ .$$

On the other hand, for every n, the SES in $\mathcal{A}$

$$0 \longrightarrow B_p(P,d^h)_n \longrightarrow Z_p(P,d^h)_n \longrightarrow H_p(P,d^h)_n \longrightarrow 0$$

is split, because $H_p(P,d^h)_n$ is projective. Therefore, $Z_p(P,d^h)_n \cong B_p(P,d^h)_n \oplus H_p(P,d^h)_n$. We conclude that

$$Z_p(P,d^h)_\bullet \to Z_p(A)$$

is a projective resolution.

Similarly, since $Z_p(P, d^h)_\bullet \to Z_p(A)$ is a projective resolution for every p, we apply the horseshoe lemma to the commutative diagram with exact columns

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
\cdots & \longrightarrow & Z_p(P, d^h)_1 & \longrightarrow & Z_p(P, d^h)_0 & \longrightarrow & \ker(d_p^A) = Z_p(A) \\
 & & & & & & \downarrow{\scriptstyle i} \\
 & & & & & & A_p \\
 & & & & & & \downarrow{\scriptstyle d_p^A} \\
\cdots & \longrightarrow & B_{p-1}(P, d^h)_1 & \longrightarrow & B_{p-1}(P, d^h)_0 & \longrightarrow & \mathrm{Im}(d_p) = B_{p-1}(A) \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}$$

A similar reasoning allows us to conclude.

(b) Provided that $\mathcal{A}$ is a small abelian category, we may assume by *Freyd-Mitchell's embedding theorem* that $\mathcal{A}$ is the category of R-modules for some ring R. We apply a similar reasoning as in the proof of theorem 2.7.2.

Consider $A_\bullet = A_{\bullet,0}$ viewed as double complex concentrated in degree 0. We start with the CE-resolution $P_{\bullet,\bullet} \to A_\bullet$ and consider the augmented double complex C' by adding the shifted double complex $A[-1]_\bullet$ in the row $q = -1$.

$$\begin{array}{ccccccccccc}
\vdots & & & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & & & {\scriptstyle d^h}\downarrow & & \downarrow & & \downarrow & & \\
P_{0,0} & \longleftarrow & \cdots & \longleftarrow & P_{p-1,0} & \overset{d^r}{\longleftarrow} & P_{p,0} & \longleftarrow & P_{p+1,0} & \longleftarrow & \cdots \\
{\scriptstyle \epsilon_0}\downarrow & & & & {\scriptstyle \epsilon_{p+1}}\downarrow & & \downarrow{\scriptstyle \epsilon_p} & & \downarrow{\scriptstyle \epsilon_{p-1}} & & \\
A_0 & \longleftarrow & \ldots & \longleftarrow & A_{p-1} & \longleftarrow & A_p & \longleftarrow & A_{p+1} & \longleftarrow & \ldots
\end{array}$$

We notice that $\epsilon : \mathrm{Tot}^{\oplus}(P_{\bullet,\bullet}) \to A$ has mapping cone

$$\mathrm{Cone}(\epsilon) = \mathrm{Tot}(C').$$

Using corollary 1.5.4, ϵ is a quasi-isomorphism if and only if $\mathrm{Cone}(\epsilon)$ is exact. But by the *Acyclic assembly lemma* (Lemma 2.7.3), since

C' is a right half plane complex with exact columns (since there are projective resolutions by part (a)) we conclude.

2. We construct the induced map $\tilde{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ as follows. We begin by constructing chain maps $\tilde{f} : P_{p,\bullet} \to Q_{p,\bullet}$ between the p-th columns of $P_{\bullet,\bullet}$ and $Q_{\bullet,\bullet}$ then verify that they assemble to a double chain complex map $\tilde{f}$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_p(A) & \xrightarrow{i} & A_p & \xrightarrow{d_p} & B_{p-1}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & Z_p(B) & \xrightarrow[i']{} & B_p & \xrightarrow[d_p]{} & B_{p-1}(B) & \longrightarrow & 0
\end{array}$$

where $g = f_p\big|_{Z_p(A)}$ and $h = f_{p-1}\big|_{B_{p-1}(A)}$. By the *Comparison theorem* (Theorem 2.2.6), the maps h and g can be lifted to chain complex maps (red arrows in the diagram below).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_p(P,d^h)_\bullet & \longrightarrow & P_{p,\bullet} & \xrightarrow{d_p^h} & B_{p-1}(P,d^h) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle H_\bullet} & & & & \downarrow{\scriptstyle G_\bullet} & & \\
0 & \longrightarrow & Z_p(Q,d^h) & \longrightarrow & Q_{p,\bullet} & \xrightarrow[d_p^h]{} & B_{p-1}(Q,d^h) & \longrightarrow & 0
\end{array}$$

We build a lift F_p of f_p out of $H_\bullet$ and $G_\bullet$. Recall that since $Z_p(P,d^h)_\bullet$ and $B_p(P,d^h)_\bullet$ are projective resolutions, the rows of the commutative diagram above are split exact so that

$$P_{p,\bullet} = Z_p(P,d^h) \oplus B_{p-1}(P,d^h) \text{ and } Q_{p,\bullet} = Z_p(Q,d^h) \oplus B_{p-1}(Q,d^h).$$

Since se work in the q direction, we will alleviate notations by writing

$$Z^P := Z_p(P,d^h) \text{ and } Z^Q := Z_p(Q,d^h)$$

$$B^P := B_{p-1}(P,d^h) \text{ and } B^Q := B_{p-1}(Q,d^h)$$

Claim 0.1. *We can construct a maps $\gamma_q : B_q^P \to Z_q^Q$ so that the following maps assemble to a chain map $F_\bullet := \{F_q\}_q$*

$$F_q := \begin{pmatrix} H_q & \gamma_q \\ 0 & G_q \end{pmatrix} : Z_q^P \oplus B_q^P \to Z_q^Q \oplus B_q^Q.$$

In other words, for any $(z,b) \in Z_q^P \oplus B_q^P$,

$$F_q(z,b) = (H_q(z) + \gamma_q(b), G_q(b)).$$

Sketch of proof. The proof is similar to the inductive construction of γ_n in Theorem 2.4.6. Simply replace P'_n with Z^P_n, P''_n with B^Q_n and the maps F'_n with H_n, F''_n with G_n. $\triangle$

Uisng the claim, we get a collection of maps $F_p : P_{p,\bullet} \to Q_{p,\bullet}$. It remains to prove that we can assemble them to a double chain complex map. Recall that we have shown in Exercise 10.1.(a) that $\epsilon_p : P_{p,\bullet} \to A_p$ and $\eta_p : Q_{p,\bullet} \to B_p$ are projective resolutions, this yields the following commutative diagram,

$$
\begin{array}{ccccccccccc}
\cdots \to & Z_p(P,d^h)_\bullet & \to & P_{p,\bullet} & \xrightarrow{d^h_p} & \overbrace{B_{p-1}(P,d^h)_\bullet}^{=Z_{p-1}(P,d^h)_\bullet} & \to & P_{p-1,\bullet} & \to & Z_{p-2}(P,d^h)_\bullet & \to \cdots \\
& {\scriptstyle H_\bullet}\downarrow & & \downarrow{\scriptstyle F_p} & & \downarrow{\scriptstyle G_\bullet} & & \downarrow{\scriptstyle F_{p-1}} & & \downarrow{\scriptstyle H_\bullet} & \\
\cdots \to & Z_p(Q,d^h) & \to & Q_{p,\bullet} & \xrightarrow[d^h_p]{} & \underbrace{B_{p-1}(Q,d^h)_\bullet}_{=Z_{p-1}(P,d^h)_\bullet} & \to & Q_{p-1,\bullet} & \to & Z_{p-2}(Q,d^h)_\bullet & \to \cdots
\end{array}
$$

This allows to conclude that $\tilde{f} := \{F_p\}_p : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ is a double chain complex map that lifts $f : P \to Q$.

3. (a) Suppose that A is concentrated in degree 0. Then the CE-resolution of A is supported on the single column $p = 0$. In particular,

$$\mathbb{L}_i F(A) = H_i(\mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet}))) = H_i(F(P_{0,\bullet})) = L_i F(A_0).$$

where the last equality follows from the fact that $P_{0,\bullet} \to A_0$ is a projective resolution.

(b) Let $A_\bullet \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$. On the one hand for all i, by definition

$$\mathbb{L}_i F(A_\bullet) = H_i(\mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet}))$$

where $P_{\bullet,\bullet} \to A_\bullet$ is a CE-resolution of $A_\bullet$. On the other hand,

$$L_i H_0 F(A_\bullet) = H^h_i(H^v_0 F(P_{\bullet,\bullet})) \overset{\clubsuit}{=} H^h_i(F H^v_0(P_{\bullet,\bullet})).$$

The equality $\clubsuit$ follows from the fact that F is right exact and $P_{\bullet,\bullet} \to A_\bullet$ is a resolution. Indeed this implies that $H_0(F(P_{\bullet,\bullet})) \cong F(A_\bullet)$. On the other hand, $H_0(P_{\bullet,\bullet}) \cong A_\bullet$. We conclude by unicity of the cokernel.

Henceforth, we write $P := P_{\bullet,\bullet}$.

Claim 0.2. $\mathrm{Tot}^{\oplus}(P)$ *is chain homotopy equivalent to* $H_0(P)$.

Proof. Consider the spectral sequence associated to P

$$^{I}E^2_{p,q} = H^h_p H^v_q(P) \implies H_{p+q}(\mathrm{Tot}^{\oplus}(P).$$

Since P is a CE-resolution, it has exact columns and in particular, $H^h_p H^v_q(P) = 0$ for all $q > 0$. Therefore page 2 of the above spectral sequence collapses on the single row $q = 0$ and we conclude that $\forall i \geq 0$

$$H^h_i H^v_0(P) = {}^{I}E^2_{i,0} = {}^{I}E^{\infty}_{i,0} \cong H_i(\mathrm{Tot}^{\oplus}(P)).$$

This shows that $\mathrm{Tot}^{\oplus}(P)$ is quasi-isomorphic to $H_0(P)$. But since $\mathrm{Tot}^{\oplus}(P)$ and $H_0(P)$ are chain complexes of pointwise projective objects, there are in fact chain homotopy equivalent (see for example Lemma 10.4.6).

△

It follows from the claim and the fact that F is an additive functor (as it is right exact) that $F(H_0(P))$ and $F(\mathrm{Tot}^{\oplus}(P)) = \mathrm{Tot}^{\oplus}(F(P))$ are chain homotopy equivalent. This latter conclusion yields the desired isomorphism

$$\begin{aligned} L_i H_0 F(A) &= H_i(F H_0(P)) \\ &\cong H_i(\mathrm{Tot}^{\oplus}(F(P)) \\ &= \mathbb{L}^i F(A). \end{aligned}$$

(c) Let $A \in \mathrm{Ch}(\mathcal{A})$ and let $P_{\bullet,\bullet}$ be a CE-resolution of A. Consider the shifted complex $A[n]$. Then the double complex obtained by shifting the columns of $P_{\bullet,\bullet}$, denoted $P[n,0]_{\bullet,\bullet}$ is a CE-resolution of $A[n]$.

On the other hand, note that for all k

$$\begin{aligned} \left(\mathrm{Tot}^{\oplus}(P[n,0])\right)_k &= \bigoplus_{i+j=k} P[n,0]_{i,j} \\ &= \bigoplus_{i+j=k} P_{i+n,j} \\ &= \bigoplus_{i'+j=k+n} P_{i',j} \\ &= \left((\mathrm{Tot}^{\oplus} P)[n]\right)_k . \end{aligned}$$

We conclude by dimension shifting

$$\mathbb{L}_i F(A[n]) = H_i\left(\mathrm{Tot}^{\oplus} F(P[n,0])\right) = H_i\left((\mathrm{Tot}^{\oplus} P)[n]\right) = H_{i+n}\left(\mathrm{Tot}^{\oplus} P\right) = \mathbb{L}_{i+n} F(A).$$

4. Consider the short short exact sequence associated to the cone A

$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow A_1[-1] \longrightarrow 0 \ .$$

Using Lemma 5.7.5, there is a long-exact sequence

$$\cdots \longrightarrow \mathbb{L}_{i+1}F(A_1[-1]) \longrightarrow \mathbb{L}_iF(A_0) \longrightarrow \mathbb{L}_iF(A) \longrightarrow \mathbb{L}_iF(A_1[-1]) \longrightarrow \cdots$$

But, by Exercises 10.3.c then 10.3.a, for any integer k,

$$\mathbb{L}_{k+1}F(A_1[-1]) = \mathbb{L}_kF(A_1) = L_kF(A_1).$$

Similarly

$$\mathbb{L}_kF(A_0) = L_kF(A_0).$$

Substituting these in the above long exact sequence yields

$$\cdots \longrightarrow \mathbb{L}_{i+1}F(A) \longrightarrow L_iF(A_1) \longrightarrow L_iF(A_0) \longrightarrow \mathbb{L}_iF(A) \longrightarrow \cdots$$

5. For the morphism of rings $f : X \to Y$ and the $\mathcal{O}_X$-module $\mathcal{F}$, consider the associated Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^qf_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).$$

   (a) The condition that $R^qf_*\mathcal{F} = 0$ for every $q > 0$ implies that page 2 of this spectral sequence is supported in the single row $q = 0$.

	$\vdots$	$\vdots$	$\vdots$	
1	0	0	0	
0	$H^0(Y, R^0f_*\mathcal{F})$	$H^1(Y, R^0f_*\mathcal{F})$	$H^2(Y, R^0f_*\mathcal{F})$	$\ldots$
	0	1	2	

Using bounded convergence, we conclude that for every $p \geq 0$

$$H^p(X, \mathcal{F}) \cong E_\infty^{p,0} = E_2^{p,0} = H^p(Y, f_*\mathcal{F}).$$

Remark 0.3. In algebraic geometry, the condition that $R^qf_*\mathcal{F} = 0$ holds for example when f is affine.

(b)

Remark 0.4. We correct the assumptions in the statement of the exercise. We assume that $H^p(Y, R^q f_* \mathcal{F}) = 0$ for all $p > 0$. In algebraic geometry, these condition is satisfied for Y an affine scheme for example.

The condition that $H^p(Y, R^q f_* \mathcal{F}) = 0$ for every $p > 0$ implies that page 2 of the Leray spectral sequence is supported on the single column $p = 0$.

	$\vdots$	$\vdots$	$\vdots$	
2	$H^0(Y, R^2 f_* \mathcal{F})$	$\vdots$	$\vdots$	
1	$H^0(Y, R^1 f_* \mathcal{F})$	0	0	
0	$H^0(Y, R^0 f_* \mathcal{F})$	0	0	$\dots$
	0	1	2	

Since at each pages, the differentials going in and out of the complexes in the single column have either 0 source or target, we conclude that for every $q \geq 0$, by bounded convergence

$$H^q(X, \mathcal{F}) \cong E_\infty^{0,q} = E_2^{0,q} = H^0(Y, R^q f_* \mathcal{F}).$$